

# THE NORMAL COORDINATE TRANSFORMATION OF A LINEAR SYSTEM WITH AN ARBITRARY LOSS FUNCTION

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**Introduction.** The normal coordinate transformation in classical dynamics is well known to be restricted to conservative systems; that is to say, systems involving potential and kinetic energy but no loss. This situation is easily understood if we remind ourselves that a nonsingular congruent transformation of the pertinent variables can simultaneously reduce no more than two quadratic forms to sums of squares, and this only if at least one of them is nonsingular and positive (or negative) definite.

Trivial exceptions occur if a third quadratic form (which in a dynamic system may be the loss function) is linearly dependent upon the other two, or if the principal axes of its associated quadric surface are coincident with those of one of the other two forms. It is likewise obvious that the normal coordinate transformation is possible for systems in which the two quadratic forms involved are potential energy and loss or kinetic energy and loss (as in the analogous electric RC or RL networks). However, it has so far not been shown how a normal coordinate transformation can generally be accomplished for a dynamic system with arbitrary loss. This is our objective in the present discussion.

**1. Equilibrium equations in "mixed" form and their transformation to normal coordinates.** Since it is mathematically impossible to reduce simultaneously more than two quadratic forms to sums of squares, it is clear that, to accomplish our objective, we must find a way to represent the equilibrium of an arbitrary dynamic system with loss, in terms of only two quadratic forms. Through a proper choice of variables this objective can be achieved; and it turns out that one of the quadratic forms represents all of the stored energy (kinetic and potential) while the other one is the total instantaneous rate of energy dissipation. Although in the following detailed discussion we shall use the linear passive electrical circuit as a vehicle for expression of pertinent relationships, it is clear that the results apply as well to any analogous linear system.

In the usual approach, equilibrium conditions for an electrical network are expressed either in terms of loop current or node-pair voltage variables. To achieve our objective we must use both loop currents and node-pair voltages as variables. Although there is nothing new about establishing equilibrium equations on such a "mixed" basis, our particular approach is chosen in a special manner that is at the same time completely general and particularly suited to accommodate a commonly encountered practical situation.

We begin by defining the branches of our network as consisting of series combinations of  $R$  and  $L$  or parallel combinations of  $G$  and  $C$ . Letting the differential operator  $d/dt$  be denoted by  $p$ , the self and mutual inductances of branches by  $l_{ks}$ , branch resistances by  $r_k$ , capacitances by  $c_k$  and conductances by  $g_k$ , we define the branch operators

$$z_{ks} = l_{ks}p + r_k \quad (1)$$

and

$$y_k = c_k p + g_k. \quad (2)$$

Pure resistive branches involve degenerate forms of either of these; therefore no loss in generality is involved by this special branch designation while an important practical situation is thus more readily accommodated.

Following a common practice\* we number the *RL* branches consecutively from 1 to  $\lambda$  and the *GC* branches from  $\lambda + 1$  to  $\lambda + \sigma$ . Voltage drops in the *RL* branches are elements in a column matrix  $v_\lambda$  and currents in the *GC* branches are those in the column matrix  $j_\sigma$ . Then, denoting a matrix of order  $\lambda$  with the elements  $z_{ks}$  by  $z_\lambda$  and a diagonal matrix of order  $\sigma$  with the elements  $y_k$  by  $y_\sigma$ , we have

$$v_\lambda = z_\lambda j_\lambda \quad \text{and} \quad j_\sigma = y_\sigma v_\sigma \quad (3)$$

in which  $j_\lambda$  is a column matrix representing currents in the *RL* branches and  $v_\sigma$  is one representing voltage drops across the *GC* branches.

The columns of the tie-set matrix  $\beta$  and of the cut-set† matrix  $\alpha$  are partitioned into groups of  $\lambda$  and  $\sigma$  and the resulting submatrices identified by subscripts indicating their row and column structure in the usual manner. Voltage sources acting around loops and current sources acting across node pairs are elements of column matrices  $e_s$  and  $i_s$  respectively. The Kirchhoff voltage and current laws are then expressed by the matrix equations

$$\beta_{l\lambda} v_\lambda + \beta_{l\sigma} v_\sigma = e_s \quad (4)$$

and

$$\alpha_{n\lambda} j_\lambda + \alpha_{n\sigma} j_\sigma = i_s \quad (5)$$

As usual we have

$$j_\lambda = (\beta_{l\lambda})_t i \quad \text{and} \quad v_\sigma = (\alpha_{n\sigma})_t e \quad (6)$$

in which the subscript  $t$  indicates the transposed matrix and the column matrices  $i$  and  $e$  contain the loop-current and node-pair voltage variables defined by the chosen tie-set and cut-set matrices in the familiar manner.

Substituting into 4 and 5 from 1, 2 and 3, and also making use of 6, we get the following equilibrium equations on a mixed basis

$$\beta_{l\lambda} z_\lambda (\beta_{l\lambda})_t i + \beta_{l\sigma} (\alpha_{n\sigma})_t e = e_s \quad (7)$$

$$\alpha_{n\lambda} (\beta_{l\lambda})_t i + \alpha_{n\sigma} y_\sigma (\alpha_{n\sigma})_t e = i_s \quad (8)$$

It is well known, for a given network graph, that the rows of a tie-set matrix are orthogonal to the rows of a cut-set matrix. Hence we have either

\* See "Introductory Circuit Theory" by E. A. Guillemin, John Wiley 1953, Ch. X arts. 2, 3, 4; pp. 491-502.

† loc. cit. pp. 496-499.

$$\beta_{l\lambda}(\alpha_{n\lambda})_t + \beta_{l\sigma}(\alpha_{n\sigma})_t = 0 \quad (9)$$

or

$$\alpha_{n\lambda}(\beta_{l\lambda})_t + \alpha_{n\sigma}(\beta_{l\sigma})_t = 0 \quad (10)$$

If we let

$$\gamma_{ln} = \beta_{l\sigma}(\alpha_{n\sigma})_t = -\beta_{l\lambda}(\alpha_{n\lambda})_t \quad (11)$$

$$-(\gamma_{ln})_t = \alpha_{n\lambda}(\beta_{l\lambda})_t = -\alpha_{n\sigma}(\beta_{l\sigma})_t \quad (12)$$

then the equilibrium Eqs. 7 and 8 may be written

$$\beta_{l\lambda}z_{\lambda}(\beta_{l\lambda})_t \dot{i} + \gamma_{ln}e = e_s \quad (13)$$

$$-(\gamma_{ln})_t \dot{i} + \alpha_{n\sigma}y_{\sigma}(\alpha_{n\sigma})_t e = \dot{i}_s \quad (14)$$

Noting Eqs. 1 and 2, and introducing the parameter matrices\*

$$L = \beta_{l\lambda}[l_{sk}](\beta_{l\lambda})_t; \quad R = \beta_{l\lambda}[r_k](\beta_{l\lambda})_t \quad (15)$$

$$C = \alpha_{n\sigma}[c_k](\alpha_{n\sigma})_t; \quad G = \alpha_{n\sigma}[g_k](\alpha_{n\sigma})_t \quad (16)$$

we can write 13 and 14 in the form

$$\left\{ \left[ \begin{array}{c|c} L & 0 \\ \hline 0 & C \end{array} \right] p + \left[ \begin{array}{c|c} R & \gamma_{ln} \\ \hline -\gamma_{nl} & G \end{array} \right] \right\} \times \left[ \begin{array}{c} i \\ e \end{array} \right] = \left[ \begin{array}{c} e_s \\ \dot{i}_s \end{array} \right] \quad (17)$$

in which we have denoted the transpose of  $\gamma_{ln}$  by  $\gamma_{nl}$  for convenience. Although, this submatrix depends only upon network topology, it seems more reasonable to associate it with the loss parameter matrices  $R$  and  $G$  because the differential operator  $p$  is associated with  $L$  and  $C$ .

The reason for choosing the mixed basis for expressing equilibrium is now clear since the form of matrix Eq. 17 is essentially that of a system characterized by two energy functions.

In contemplating a normal coordinate transformation we are reminded at this point that a linear transformation applied to the variables in a quadratic form results in a congruent transformation of its matrix. A normal coordinate transformation must, therefore, be accomplished by a congruent transformation of the matrices in Eq. 17. However, the simultaneous diagonalization of these two matrices by such a transformation requires both to be symmetrical (besides stipulating that one of them be definite and nonsingular). This condition is not met by the matrix containing loss elements, which in partitioned form has skew-symmetric character.

In this regard we should become aware of the fact that it is illogical to expect that the desired coordinate transformation will be accomplished by a real transformation. Since the latent roots involved are natural frequencies of our network,

\* The quantities  $l_{sk}$ ,  $r_k$ , etc. in square brackets denote the branch parameter matrices having these elements. See loc. cit. pp. 492-495 and allow for the difference in the way resistances are dealt with here.

and these are surely complex in any general  $RLC$  situation, we expect that the pertinent congruent transformation will involve a matrix with complex elements.

In anticipation of these things it is, therefore, not inconsistent to do a little multiplying by the operator  $j = \sqrt{-1}$  and put Eq. 17 into the modified but equivalent form

$$\left\{ \begin{bmatrix} Lp & | & 0 \\ \hline 0 & | & Cp \end{bmatrix} + \begin{bmatrix} R & | & j\gamma_{ln} \\ \hline j\gamma_{nl} & | & G \end{bmatrix} \right\} \times \begin{bmatrix} ji \\ \hline e \end{bmatrix} = \begin{bmatrix} je_s \\ \hline i_s \end{bmatrix} \quad (18)$$

Regarding the required nonsingular character of the first matrix, we will assume that if any linear constraint relations exist among the loop currents or node-pair voltages, these have been used to eliminate superfluous variables so that all are now dynamically independent.\* If we are considering a passive system then the positive definiteness of the quadratic form for the stored energy is assured. Hence the problem of carrying out a normal coordinate transformation upon the equilibrium equation 18 is now routine.

We may carry it out in two steps by first transforming the matrix with  $L$  and  $C$  congruently to its canonic form and then subjecting the resulting loss matrix to an orthogonal transformation with its modal matrix, thus converting it to the diagonal form and leaving the canonic matrix unaltered. These two steps may be combined into a single congruent transformation with a matrix  $A$  characterizing the coordinate transformation indicated by the equations

$$\begin{bmatrix} ji \\ \hline e \end{bmatrix} = A \times \begin{bmatrix} ji' \\ \hline e' \end{bmatrix} \text{ and } \begin{bmatrix} je_s \\ \hline i_s \end{bmatrix} = A^{-1} \times \begin{bmatrix} je'_s \\ \hline i'_s \end{bmatrix} \quad (19)$$

which convert the equilibrium equations 18 into

$$A_t \times \left\{ \begin{bmatrix} Lp & | & 0 \\ \hline 0 & | & Cp \end{bmatrix} + \begin{bmatrix} R & | & j\gamma_{ln} \\ \hline j\gamma_{nl} & | & G \end{bmatrix} \right\} \times A \times \begin{bmatrix} ji' \\ \hline e' \end{bmatrix} = \begin{bmatrix} je'_s \\ \hline i'_s \end{bmatrix} \quad (20)$$

with

$$A_t \times \begin{bmatrix} Lp & | & 0 \\ \hline 0 & | & Cp \end{bmatrix} \times A = \begin{bmatrix} u_l & | & 0 \\ \hline 0 & | & u_n \end{bmatrix} \quad (21)$$

in which  $u_l$  and  $u_n$  are unit matrices of order  $l$  and  $n$  respectively, and

$$A_t \times \begin{bmatrix} R & | & j\gamma_{ln} \\ \hline j\gamma_{nl} & | & G \end{bmatrix} \times A = D = \begin{bmatrix} d_1 & 0 & 0 & \cdot & 0 \\ 0 & d_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & d_b \end{bmatrix}. \quad (22)$$

Diagonal elements  $d_1 \cdots d_b$  are the complex natural frequencies of the system. Thus Eq. 20 expresses equilibrium in the normal coordinates to which the primed variables refer, and Eqs. 19 are the pertinent transformation relations for the variables and sources.

\* Incidentally, the matrix involving  $R$  and  $G$  is then in general also nonsingular.

Regarding energy relations we observe that premultiplication on both sides of Eq. 18 by  $[-j\dot{i}_t : e_t]$ , the transposed conjugate of the column matrix for the variables in 18, gives

$$i_t L p i + e_t C p e + i_t R i + e_t G e = i_t e_s + e_t i_s \quad (23)$$

because

$$-e_t \gamma_{nt} \dot{i} + i_t \gamma_{tn} e = 0 \quad (24)$$

as is evident from the fact that these terms are each other's transpose, but are matrices of order one. The right-hand side of Eq. 23 represents power supplied by the sources; the first two terms on the left represent rate of energy storage, and the remaining two terms are rate of energy dissipation in the lossy elements.

Analogous energy relations are obtained for the transformed Eq. 20 through premultiplication on both sides by  $[-j\dot{i}'_t : e'_t]$ . Use of Eqs. 21 and 22 then yields a conservation of energy expression analogous to Eq. 23 for the system in terms of its normal coordinate representation.

**2. Related impedance and admittance transformations.** In the equilibrium Eqs. 18 we make the familiar substitutions

$$i = I e^{st}, \quad e = E e^{st}, \quad i_s = I_s e^{st}, \quad e_s = E_s e^{st} \quad (25)$$

and introduce the further notation

$$U = \begin{bmatrix} jE_s \\ I_s \end{bmatrix}; \quad V = \begin{bmatrix} jI \\ E \end{bmatrix}, \quad (26)$$

$$W = \begin{bmatrix} Ls + R & j\gamma_{tn} \\ j\gamma_{nt} & Cs + G \end{bmatrix}, \quad (27)$$

whereupon these equations are given by

$$W \times V = U \quad (28)$$

and Eqs. 21 and 22 yield

$$A_t \times W \times A = W_d = \begin{bmatrix} (d_1 + s) & 0 & \cdots & 0 \\ 0 & (d_2 + s) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (d_b + s) \end{bmatrix}. \quad (29)$$

Solution of Eq. 28 involves the inverse matrix

$$W^{-1} = A \times W_d^{-1} \times A_t = X \quad (30)$$

If we partition this inverse matrix as indicated in

$$X = \begin{bmatrix} X_{tt} & jX_{tn} \\ \cdots & \cdots \\ -jX_{nt} & X_{nn} \end{bmatrix} \quad (31)$$

then the inverse of 28 with 26 substituted gives

$$X_{il}E_s + X_{ln}I_s = I \quad (32)$$

$$X_{nl}E_s + X_{nn}I_s = E$$

Observe that the elements in  $X_{il}$  are short-circuit driving-point and transfer admittances; those in  $X_{nn}$  are open-circuit driving-point and transfer impedances; and those in  $X_{ln}$  and  $X_{nl}$  are dimensionless transfer functions.  $X$  may, therefore, appropriately be referred to as an *immittance matrix*.

If we write for the matrix  $A$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1b} \\ a_{21} & a_{22} & \cdots & a_{2b} \\ \cdots & \cdots & \cdots & \cdots \\ a_{b1} & a_{b2} & \cdots & a_{bb} \end{bmatrix} \quad (33)$$

then Eqs. 29 and 30 show that a typical element  $x_{ik}$  of the matrix  $X$  is given by

$$x_{ik} = \sum_{v=1}^b \frac{k_v}{s + d_v} \quad (34)$$

with

$$k_v = a_{iv}a_{kv}. \quad (35)$$

Eq. 34 we recognize as a partial fraction expansion of the immittance  $x_{ik}$  which is a rational function of the complex frequency  $s$ . The residues  $k_v$  in this expansion are related to elements in the  $i^{\text{th}}$  and  $k^{\text{th}}$  rows of transformation matrix  $A$  by the simple expression 35. This situation is ideally suited to the synthesis problem, for it enables one to construct the matrix  $A$  to suit the requirements of a stated  $x_{ik}$  function. For a driving-point function ( $i = k$ ) the residues fix only a single row of  $A$ ; for a transfer function only the products of respective elements in two rows are fixed. The remainder of the matrix  $A$  is then free to choose so as to yield parameter matrices fulfilling realizability conditions.

**3. A transformation having more general applicability to synthesis.** Use of the present results in this novel approach to the synthesis problem is discussed in a separate paper, and so we shall not pursue their implications in further detail here. Rather we wish to point out that, to achieve results suitable for this kind of an attack upon the synthesis problem, we do not necessarily have to accomplish a normal coordinate transformation, but only the simultaneous diagonalization of the parameter matrices in the equilibrium Eqs. 17, which is less restrictive and hence allows greater latitude in the conditions under which it can be carried out.

If diagonalization of the matrices in 17 is our only objective, then we are not limited to a congruent transformation but can pre- and post-multiply by any nonsingular matrices that accomplish the desired end. In this situation the posi-

tive definiteness of neither of the two matrices nor their symmetry is required, and so we can include equilibrium equations for linear systems that are active and/or nonbilateral if we wish.

For the moment, however, suppose we assume that any active or nonbilateral elements are resistive in character so that the matrix with  $L$  and  $C$  in 17 is still symmetrical and pertinent to a positive definite quadratic form. Assume further that any linear constraint relations among the variables have been used to eliminate dynamically superfluous ones so that this quadratic form is non-singular. Then we can straight-forwardly construct a matrix  $P$  which congruently transforms the first matrix in 17 to its canonic form and the second to some other dissymmetrical matrix,  $B$ , thus

$$P_t \times \left[ \begin{array}{c|c} L & 0 \\ \hline 0 & C \end{array} \right] \times P = U_b = \text{unit matrix of order } b \quad (36)$$

and

$$P_t \times \left[ \begin{array}{c|c} R & \gamma_{ln} \\ \hline -\gamma_{nl} & G \end{array} \right] \times P = B = \text{dissymmetrical matrix of order } b \text{ and rank } b. \quad (37)$$

A co-linear transformation of  $B$  with an appropriate matrix  $Q$  now yields a diagonal matrix  $D$  (whose elements are the latent roots of  $B$ ) and leaves the canonic matrix 36 unchanged. This co-linear transformation of  $B$  reads

$$Q^{-1} \times B \times Q = D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdot & \cdots & d_b \end{bmatrix} \quad (38)$$

in which the diagonal elements, as in Eq. 22, are the complex natural frequencies of our linear system, and the matrix  $Q$  is in general complex.

The variables in Eq. 17 are thus transformed as indicated by the relations

$$\begin{bmatrix} i \\ \cdots \\ e \end{bmatrix} = PQ \begin{bmatrix} i' \\ \cdots \\ e' \end{bmatrix} \text{ and } \begin{bmatrix} e_s \\ \cdots \\ i_s \end{bmatrix} = P_t^{-1}Q \begin{bmatrix} e'_s \\ \cdots \\ i'_s \end{bmatrix} \quad (39)$$

although these are now not of any particular interest.

For the substitutions 25 and the notation

$$U = \begin{bmatrix} E_s \\ \cdots \\ I_s \end{bmatrix}; \quad V = \begin{bmatrix} I \\ \cdots \\ E \end{bmatrix} \quad (40)$$

and

$$W = \left[ \begin{array}{c|c} Ls + R & \gamma_{ln} \\ \hline -\gamma_{nl} & Cs + G \end{array} \right] \quad (41)$$

which are inappreciably different from 26 and 27, we again have our equilibrium equations in the form 28. Instead of 29 and 30, however, we now find

$$Q^{-1} \times P_t \times W \times P \times Q = W_d = \begin{bmatrix} (d_1 + s) & 0 & 0 \cdots & 0 \\ 0 & (d_2 + s) & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (d_b + s) \end{bmatrix} \quad (42)$$

and

$$W^{-1} = X = PQW_d^{-1}Q^{-1}P_t. \quad (43)$$

With the matrix  $X$  partitioned as indicated in

$$X = \begin{bmatrix} X_{ll} & X_{ln} \\ \cdots & \cdots \\ X_{nl} & X_{nn} \end{bmatrix} \quad (44)$$

we again have solutions in the form of Eqs. 32; and elements of  $X$  have the representation in Eq. 34. However, in place of Eq. 35, the residues are now given by the expression

$$k_v = h_{iv}t_{kv} \quad (45)$$

in which  $h_{iv}$  and  $t_{kv}$  are elements in the matrices

$$H = P \times Q \quad \text{and} \quad T = P \times Q_t^{-1} \quad (46)$$

An approach to the synthesis of active and/or nonbilateral networks through construction of parameter matrices from given rational driving-point and transfer functions may thus be formed to follow essentially the same pattern as for passive bilateral networks.

It is important, however, to observe that the transformation of variables indicated in Eqs. 39, although resulting in the simultaneous diagonalization of the pertinent parameter matrices, is not a normal coordinate transformation and does not preserve the invariance of the associated energy forms. To further our objective so far as synthesis is concerned, we do not need the energy invariance and are only interested in achieving the diagonal forms in Eqs. 29 and 42. In a somewhat less rigid sense we might still regard this process as a normal coordinate transformation since it bears a close resemblance to it and may be given a similar physical or geometrical interpretation.

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